Voronoi Diagrams - a Survey

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Abstract. The Voronoi diagrams have had a considerable effect on different areas of interest for the development of engineering, geography, mathematics, systems and others. In this document we present a description from their beginnings, showing the formality of their properties, explaining construction algorithms and finally mentioning their relation with other technologies such as the parallel computing, the GPU's and other areas of interest. The Voronoi diagrams present a structure apparently simple at first glance that we may even observe in different phenomenon of the nature but it is important to understand the duality through other structure called Delaunay triangulation.

Keywords. Convex hull, delanuay triangulation, graphic processing unit, natural neighbor interpolation, voronoi diagrams.

1 Introduction

The human mind has forged space domination during the evolution of its being, for us it is important to understand and delimit spatial areas, either for knowledge of territorial areas, shorter searches for resources or even for the visual esthetics. Step by step, a great variety of cases with limited scope are being presented but with the passage of the time it calls the attention of specialists and results in different studies and applications until forming a research area.

The Voronoi diagrams help us to form an organized decomposition of a determined space and according to a set of point elements (Voronoi sites). This concept was used in order to provide one of the solutions for the Kaplan conjecture on the optimum packaging of spheres and Decartes employs it in the book Principles of Philosophy published in 1644 to apply the vortex theory to the functioning of the universe, where the stars would be the center of celestial vortices and at the same time the Sun would be one of them that would drag the planets through an invisible fluid. In the year of 1854 in London during a cholera outbreak, John Snow performed geographical methods in order to identify the origin of the epidemics, where the faucets represented the sites of a Voronoi diagram. However, formally those who introduce an adequate definition of this spatial structure are the mathematics Gustav Dirichlet and Georges Voronoi, who name them as Dirichlet tessellation [1] in 1850 and Voronoi diagrams [2] in 1908 respectively. Georges Voronoi discovers the duality of this structure when connecting two sites that have a common border, but it is Boris Delone in 1934 who defined this property as the Delaunay triangulation, employing the empty sphere method [3].

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The first applications go back to 1911 when Alfred Thiessen performs meteorological studies in the calculation of precipitations and in 1927 Paul Niggli reports research in crystallography. Subsequently until the present, the Voronoi diagrams and the Delanuay triangulation are employed in different areas, to mention some of them we have: astronomy, geometric calculation, construction of models, architectonic design, geology, meteorology, optimization, robotics and geographic information systems [4].

2 Definition and properties

2.1 Voronoi diagrams

We will express the Euclidean distance between two points as $p = (p_1, p_2, \dots, p_n)$ and $q = (q_1, q_2, \dots, q_n)$ as $dist(p, q)$, defined as

$$dist(p,q) = ((p_1 - q_1)^2 + (p_2 - q_2)^2 + \cdots + (p_n - q_n)^2)^{1/2} \tag{1}$$

The perpendicular line that divides the segment $\overline{pq}$ in two equal parts (perpendicular bisector) is expressed as $B(p, q)$ and defined as

$$B(p,q) = \{x \in \mathbb{R}^n \mid dist(x,p) = dist(x,q)\} \tag{2}$$

$P = \{p_1, p_2, \dots, p_m\}$ being a set of m different points on the plane (these points are called sites). We will define the Voronoi diagram of first order of P as a subdivision of the plane in m regions (see figure 1), where each region T_i is associated to a site p_i, such that any point T_i is close to p_i, the above mentioned is defined as:

$$T_i = \{x \in \mathbb{R}^n \mid dist(x,p_i) < dist(x,p_j), \forall\, i \neq j\} \tag{3}$$

The geometric structure of the diagram is formed by semi-circles and Voronoi vertices. Some important properties of the Voronoi diagram in two dimensions are listed below:

- A Voronoi edge is defined as a perpendicular bisector of the two closest sites p_i and p_j, see figure 2. The condition of the circle that contains two close sites and the edge that passes through its center should comply with the condition that no other site should be in its interior.
- A Voronoi vertex is the intersection of three edges of the diagram, the vertex also represents the center of a circle defined by three sites p_i, p_j and p_k, as long as the Voronoi diagram is regular or grade three, see figure 3. As condition of the circle generated by the three sites, it should not contain other sites in its interior.
- The regions formed by the diagram are convex polygons or non-enclosed regions.

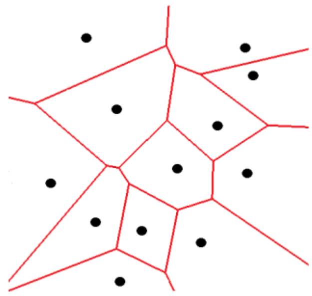
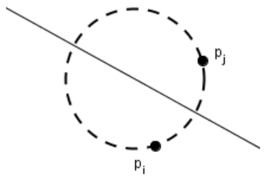


Fig. 1. Voronoi diagram in two dimensions.



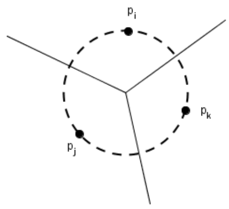
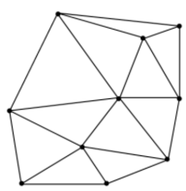


Fig. 2. Two sites. **Fig. 3.** Three sites.

2.2 Delaunay triangulation

The Delaunay triangulation is a graphic that presents the dual characteristic of the Voronoi diagram, it is only needed to unite the sites that share a common edge and complement with a convex cover where applicable. But to construct a Delaunay triangulation from the Voronoi diagram is costly which is why other techniques are used to generate it. In the figure 4 a triangulation is shown following the properties of the work of Boris Delone and the figure 5 shows the duality of both structures.

The Delaunay triangulation is used for the generation of nets, defining a method to connect an arbitrary set of points in a manner that they form a topologically valid set of triangles that does not intercept [5].



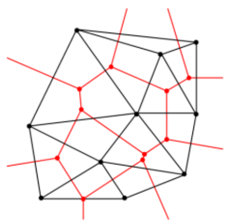


Fig. 4. Delaunay triangulation. **Fig. 5.** The duality of both structures.

The main properties of the Delaunay triangulation in two dimensions are:

- For a set of points $P = \{p_1, p_2, \dots, p_m\}$, where D is a Delaunay triangulation of P if and only if no point P is in the interior of a circle formed by the circumscribed circumference of any triangle of D, see figure 6 and 7.
- A convex cover is formed in the border points of the Delaunay triangulation.
- A Delaunay triangulation is unique.

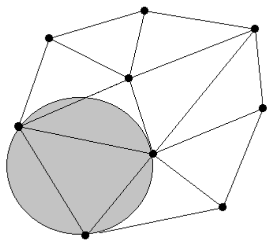


Fig. 6. Legal triangulation.

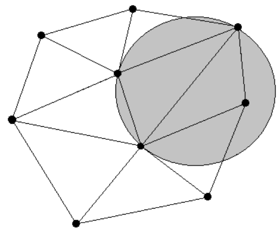


Fig. 7. Illegal triangulation.

3 Construction algorithms

Due to the extensiveness of the algorithms to generate the Voronoi diagrams the two most classic from its history will be mentioned and explained in a brief but concise manner.

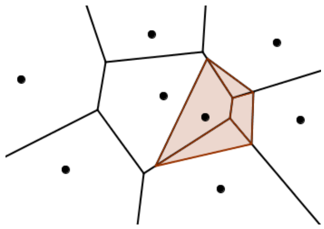
3.1 Incremental algorithm

This algorithm can be considered as one of the simplest and most intuitive procedures, one of the first works where the incremental method is used in the Voronoi diagrams was studied by Green and Sibson [6]. The idea that implements the present algorithm consists in adding one site at a time from the set of point sites P (a complexity $O(n^2)$), the figure 8 illustrates an example of the addition of a new site to the Voronoi diagram, the shaded area indicates the space occupied by the added site. For each one of the insertions the diagram is modified and the problem resolved in iterative manner. The technique of natural neighbor interpolation introduced by Sibson will be used that makes reference to the use of the second order Voronoi diagram. The cloud of sites P is subdivided in regions called T_{ij}, and now each region T_{ij} defines the geometric place of the site p_i as the closest one and the site p_j as the second closest, this is presented in the following manner:

$$T_{ij} = \{x \in \mathbb{R}^n \mid dist(x, p_i) < dist(x, p_j) < dist(x, p_k), \forall\, i \neq j \neq k\} \qquad (4)$$

The inserted site will generate a new region which is why the Voronoi diagram will have to be modified; this new space is formed with perpendicular bisectors between the close neighbor sites and creates areas proportional to the recently created site.

Sibson established that the coordinates of a natural neighbor of a point x generate a weight that is proportional to each one of their natural neighbors, generating a section of T_{xi} region. The figure 9 establishes an example, the added point x generates the area understood as T_{x1} (shaded area) formed by the closed polygon for the site 1. We also observe in the figure 8 that x has four natural neighbors (sites 1, 2, 3 and 4) and each one generates a section of region around and within x, the sum of these regions corresponds to T_x (shaded area in the figure 8).



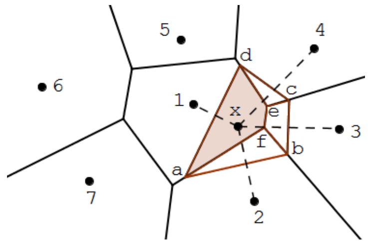


Fig. 8. A new site added. **Fig. 9.** There are four natural neighbours.

From other point of view, this method of insertion can be better implemented through its dual modality and it is called the incremental flipping algorithm. In the same manner, when a site is added, the triangulation should be reconstructed complying with the properties of the empty sphere. During the insertion a new net is generated around the new site, which is why the illegal edges have to be legalized and every new generated triangle is traversed, validating it with regards to its property of circumscribed circle. In the event that it does not comply we proceed to exchange the edge with the neighbor triangle, the procedure is illustrated in the figure 10.

Let us try to solve a real life problem, we have several post office locations in a city, see figure 11a (the small circles are the post offices). For each location we have the coordinates and we can represent them in a Euclidean space. Then the problem is visualize and find the closest post office to a given house by proximity. In the figure 11b the dots represent the post offices.

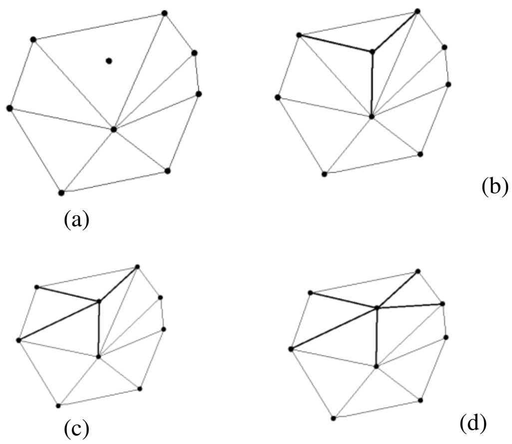


Fig. 10. a) Added a new point b) General triangulation c) First flip d) Second flip.

Now the postman must go out and make a delivery, but it would be wasteful if another post office is closer (we need a region optimal for the postman). The service region for a postman is a cell of Voronoi diagram. We know the algorithm to how construct a Voronoi diagram with a set of points in the plane. The figure 11c show us this construction and finally the figure 11d we can see the application about in the map.

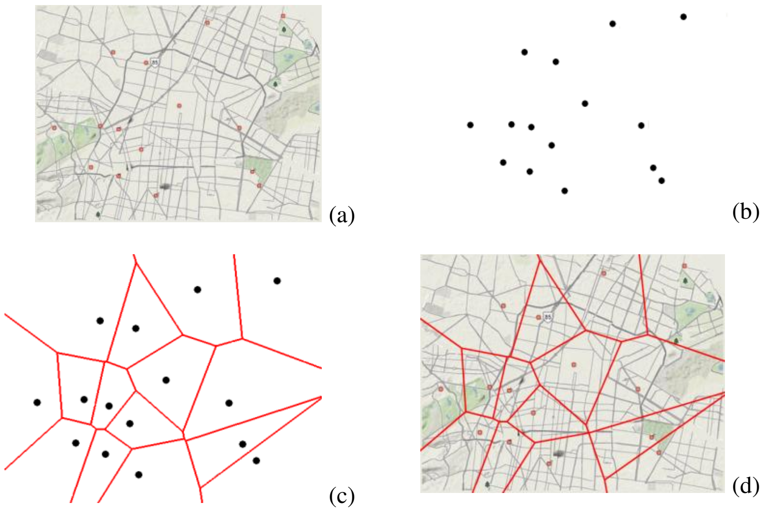


Fig. 11. a) Post offices in the map b) Sites represent the post offices c) Construction of Voronoi diagram d) Voronoi diagram in the map.

3.2 Divide and conquer algorithm

The method divide and conquer is one of the fundamental paradigms in the design of algorithms, where the idea is to divide the main problem in various simpler problems in order to subsequently find a solution based on them.

The first ones to use this technique were Shamos and Hoey [7] with a complexity O(n logn) to construct the Voronoi diagram with an optimum computer cost in the worst case scenario. The process starts with a subdivision of the set of sites of P, either horizontal or vertical, we will have two halves with approximately the same quantity of sites to which a convex hull is applied, see figure 12a. Subsequently the Voronoi diagram is constructed in a recursive manner for each subset of sites, see figure 12b.

The most important procedure is to construct a polygonal open line that would be dividing into two subsets where each straight segment represents a bisector between two points from different subsets, points that have the characteristic of forming part of the convex cover, they are border of the subsets to be united and form close neighbors. The straight segment changes trajectory each time that it intersects with a Voronoi diagram edge entering other region generated by other site and other bisector is constructed complying with the above mentioned characteristics, see figure 12c.

The Voronoi diagrams are constructed recursively in order to perform afterwards the corresponding unions of each pair of subsets until including all the sites of P.

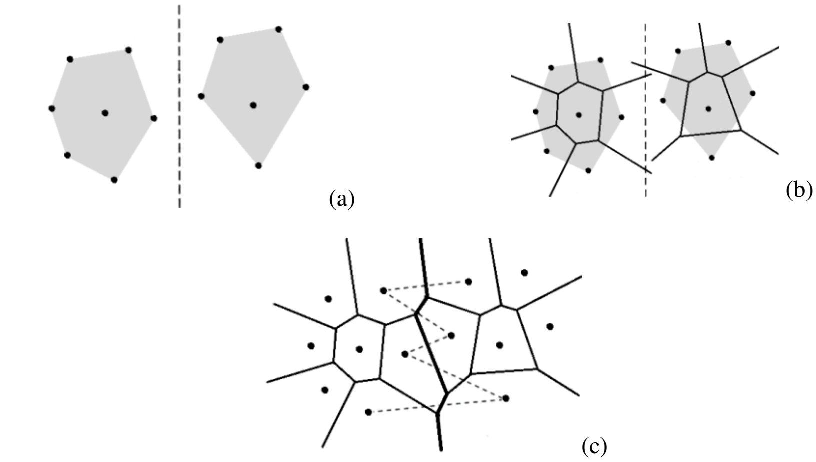


Fig. 12. Divide and conquer algorithm for Voronoi diagram.

The divide and conquer algorithm also applies to the Delaunay triangulation, in the same manner as to the Voronoi diagram, the set of points is divided in two subsets of the same size, each Delaunay triangulation is calculated separately and finally the two triangulations are united complying with the established criteria. This implementation was elaborated by Guibas and Stolfi [8], see figure 13.

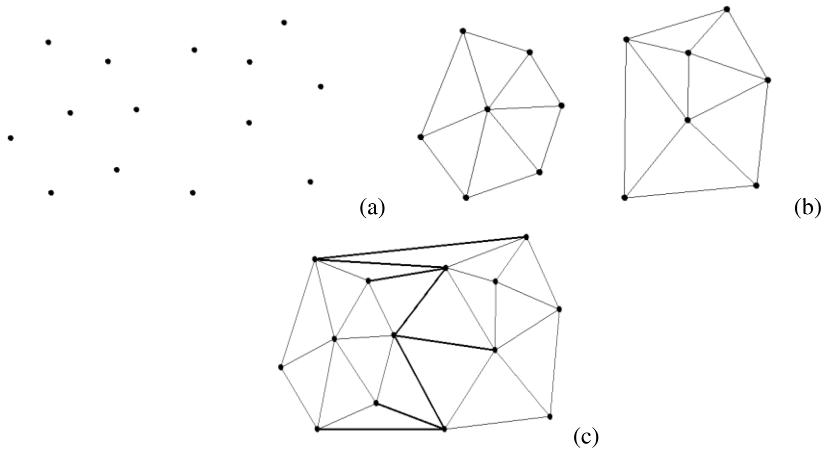


Fig. 13. Technique divide and conquer for Delaunay triangulation.

The above mentioned techniques were shown in a 2D space, even though many important applications in different disciplines require greater dimensionality, the Voronoi diagrams are also elaborated in a 3D environment. Deok-Soo Kim and his colleagues worked in 2005 on an algorithm for construction of the Voronoi diagrams in an Euclidean space using spheres in 2D, the complexity of its algorithm is of O(m n) in the worst case scenario, where m is the number of the Voronoi diagram edges and n is the number of spheres [9].

4 Voronoi diagrams in other areas

4.1 Parallel computing in voronoi diagrams

In the applications of the Voronoi diagrams where multiple calculations or operations have to be performed in real time, the need has been observed to address the analysis, design and implementation of parallel algorithms. Sequential algorithms had been developed since 1975 for the geometric computing problems but since 1980 a noticeable contribution has been presented by Chown [10] in the parallel computing for geometric problems. The work performed by Aggarwal et al in 1985 should be noted [11], where they present an efficient parallel algorithm in order to construct the Voronoi diagrams among other geometric applications.

A parallel algorithm primarily needs parallel computing in order for different parts of the algorithm to be executed by various processors in simultaneous manner and to finally unite them to obtain the desired result. The technological progress has produced machines that have become more and more powerful and generally work with more than one processor; fortunately the cost of hardware is not so elevated, which allows for the parallel processing applications to be used with greater frequency. A technology has been introduced recently that releases the load of operations of the central processing unit (CPU) and also the memory depending on the applications, these are sent to a graphic processing unit (GPU) that were used primarily for the use of videogames and graphic operations. Thanks to the great capacity of operations that the GPU's are able to perform, due to the considerable quantity of processors that they contain, they have been employed in different areas of science and engineering to accelerate the involved calculations. The concept of construction of the Voronoi diagram used with GPU was employed by Rong and Zhan Yuan in the works [12] y [13].

4.2 Parallel algorithm

The divide-and-conquer is a variant of the more general top-down programming strategy and is one of the first optimal solution allowing for efficient parallelization for constructing Voronoi diagram or the Delaunay triangulation. This structures have been proposed for computing in parallel in computational geometry. The technique divide-and-conquer try to work out a problem into two or more subproblems, every subproblem can be solve recursively in parallel. The crux is solved through a series of merges for every subproblem.

We going to speak about the Delaunay triangulation for use to give a parallelism with the divide-and-conquer strategy. In the figure 14 we appreciate the algorithm when the cloud of points is divided in two groups, every group is solve with a compute process independent (we can see the illustration of the procedure in the figure 13).

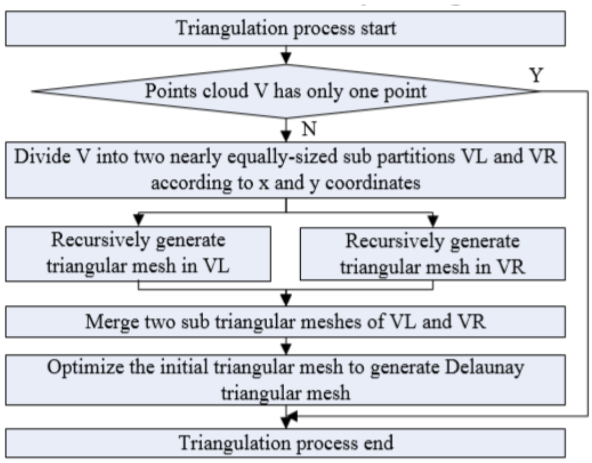


Fig. 14. Divide and conquer algorithm for Delaunay triangulation [14]

If we have computers with large numbers of process unit are capabilities to run multiple processes that solve everyone of block of triangulation, see figure 14 a and b. Merging the blocks is the critical step and this procedure can used parallel scheme for the merging phase, in the figure 13 b and c we can see the merger of two triangulation.

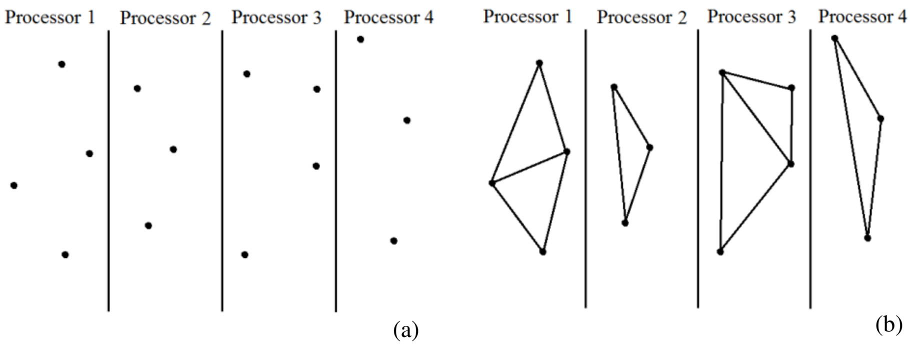


Fig. 15. A set of point going to be worked for four processors.

4.3 Voronoi diagrams and other applications

In the robotics area we have various applications such as the planning of routes of robot using Voronoi diagrams or when the robot should move avoiding collisions, which is why the trajectory of the robot should be designed. In 2007 Bhattacharya and Gavrilova worked on the planning of optimum routes using Voronoi diagrams [15]. Working with the same idea of the robots, M. Gold in 2006 proposes the use in the field of the GIS (Geographic Information Systems from its English initials) in order to avoid collisions with geographic accidents [16]. Within the GIS, the number of applications is considerable when trying to adjust the Voronoi diagrams to the geographic characteristics of the different regions of the planet. The use of classifiers in pattern recognition using the Voronoi diagrams is employed in techniques of minimum distance or the closest neighbor (that are non-parametrical rules for techniques), as used in the work of Narendra Ahuja [17]. This tool provides countless applications such as those used in image recognition and an example is the one used by Abbas Cheddad in order to obtain the extraction of characteristics of human face [18].

5 Conclusions

The Voronoi diagram is a fundamental structure in many applications of technology and is relatively simple concept with a wide gamma of applications in science and engineering. We observe a promising future in Voronoi diagram applications, through unprecedented development when use computing parallel and graphic processing unit.

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References

1. Dirichlet, G.: Uber die Reduction der positiven quadratischen Formen mit drei unbestimmten ganzen Zahlen, Journal fur die reine und angewandte Mathematik 40, no. 3, 209-227 (1850).
2. Voronoi, G.: Nouvelles applications des parametres continus a la theorie des formes quadratiques, Journal fur die reine und angewandte Mathematik 134, no. 4, 198-287 (1908).
3. Delone , B. N.: Sur la sphere vide, Bulletin of the Academy of Sciences of the U. S. S. R., no. 6, 793-800 (1934).
4. http://www.voronoi.com
5. Cofre, R.: http://cybertesis.ubiobio.cl/tesis/2003/cofre_r/html/index-frames.html
6. Green, P. J., Sibson, R. R.: Computing Dirichlet tessellations in the plane. Comput. J., 21:168–173 (1978).
7. Shamos, M. I., Hoey, D.: Closest-point problems. In Proc. 16th Annu. IEEE Sympos. Found. Comput. Sci., pages 151–162 (1975).
8. Guibas, L. J., Stolfi, J.: Primitives for the manipulation of general subdivisions and the computation of Voronoi diagrams. ACM Trans. Graph., 4(2):74– 123, Apr. (1985).
9. Kim D., Cho Y., Kim D.: Euclidean Voronoi diagram of 3D balls and its computation via tracing edges, Computer-Aided Design, Vol. 37, No. 13, pp. 1412-1424, (2005).

10. Chow, A.: Parallel algorithms for geometric problems. Ph.D. Dissertation, Computer Science Department, University of Illinois at Urbana-Champaign, (1980).
11. Aggarwal, A., Chazelle, B., Guibas, L., O'Dunlaing, C., Yap, C. K.: Parallel computational geometry, Algorithmica 3, 293-327, preliminary version in FOCS 1985, pp. 468-477 (1988).
12. Rong, G., Tan, T.: Jump Flooding in GPU with Applications to Voronoi Diagram and Distance Transform, ACM Article. Bibliometrics Data, pp 109 - 116 (2006)
13. Yuan, Z., Rong, G., Guo, X.: Generalized Voronoi Diagram Computation on GPU, Eighth International Symposium on Voronoi Diagrams in Science and Engineering, pp 75 - 82, (2011).
14. Zhe, W., Sanhong, G., Lichun, L.,: A Parallel Delaunay Algorithm Applied in Lunar Rover Stereo Vision System, Proceedings of the 2nd International Conference on Computer Science and Electronics Engineering (2013).
15. Bhattacharya, P., Gavrilova, M.: Voronoi Diagram in Optimal Path Planning, ISVD 2007, IEEE Proceedings, pp. 38-47 (2007).
16. Gold, C. M.: What is GIS and what is not? Transactions in GIS, 10(4), (2006)
17. Narendra, Ahuja: Dot Pattern Processing Using Voronoi Neighborhoods, IEEE Transactions on Pattern Analysis and Machine Intelligence, vol. 4, no. 3, pp. 336-343 (1982).
18. Abbas, C., Dzulkifli, M., Azizah, M.: Exploiting Voronoi Diagram Properties in Face Segmentation and Features Extraction. Pattern Recognition 41 (12) 3842-3859, Elsevier Science (2008).